HYPERSTABILITY

Robert Schoen and Young J. Kim

Hopkins Population Center Papers on Population
WP 93-10
(Revised, August, 1996)

This work was supported by grants R01 HD19145 and R01 HD28443 from the Center for Population Research (NICHD), and benefitted from the research assistance of Robert Weinick and support provided to the Hopkins Population Center by NICHD grant P30 HD06268 and NCRR Shared Instrumentation grant 1 S10 RR07268.

Corresponding Address: Dr. Robert Schoen
Department of Population Dynamics
Johns Hopkins University School of Hygiene and Public Health
615 N. Wolfe Street
Baltimore, MD 21205

Hyperstability

ABSTRACT

This paper describes a closed form demographic model with changing vital rates. The hyperstable model replaces the strict stable population assumption of constant rates of fertility and mortality with weaker assumptions on the pattern of net maternity. Those alternative assumptions are a fixed proportional distribution of births by age of mother, a fixed proportional distribution of net maternity, and a constant generation length.

In continuous form, we examine 6 different birth functions that span a broad range of demographic experience, illustrating cubic exponential and sinusoidal models with hypothetical data. In discrete form, assuming a fixed proportional distribution of births, we demonstrate that convergence to hyperstability follows the same strong ergodic theorem as convergence to stability. Data for the United States 1960-89 show that hyperstable estimates are close to observed Net Reproduction Rates, even though the post-1960 U.S. population is not hyperstable. Analytically, hyperstable relationships can model alternative trajectories to stationarity. The flexible and dynamic nature of hyperstable populations suggests great potential for use in demographic modeling and estimation.

Hyperstability

The stable population has long been the principal model of mathematical demography. Its strengths come from its logical closure, its ability to reflect the implications of any set of birth and death rates, and its central insight that constant vital rates lead to exponential growth and an unchanging age distribution. Yet the stable model has a great weakness---it is based on unchanging vital rates. That fixed rate requirement essentially makes the stable population a static model, unable to reflect the behavior of actual populations.

Dynamic models, with changing vital rates, are needed to enable population analysis to capture observed demographic behavior. Despite the research done to date, progress toward dynamic modeling has been impeded by a fundamental problem. With the exception of some limited, special cases, there is no known closed form relationship for the sequence of births generated by a series of changing vital rates.

This paper addresses that problem. It extends stable population theory to develop what we have termed hyperstable models, populations governed by closed form mathematical equations that can reflect a large number of birth sequences and, in each case, specify consistent sequences of vital rates. We explore the structure of hyperstable models, discuss some of the relationships that exist within them, demonstrate convergence to hyperstability, and offer several numerical illustrations.

A BRIEF LOOK AT THE LITERATURE ON CHANGING RATES

A milestone in the work on the deterministic[1] modeling of dynamic rates is the monograph by Coale (1972), which contained in-depth analyses of the relationship between changing vital rates and the birth sequence. In an analysis particularly relevant here, Coale (1972, Ch. 4) investigated the assumption that, for some constant k,

$$m(x,t) = e^{kt}\, m(x,0) \qquad (1)$$

where m(x,t) is the fertility rate at age x and time t. He found that the resultant birth sequence was approximated by

$$B(t) \approx B(0)\exp[\tfrac{1}{2}kt + \tfrac{1}{2}kt^2/T_o]$$

where B(t) is the number of births at time t and T_o is the mean age at childbearing at the time when the population's Net Reproduction Rate (NRR) equals one. Coale also presented a closer approximation involving higher powers of t, but could not find a closed form expression for B(t).

Cyclically stable populations, which result when a fixed sequence of rate schedules repeats itself indefinitely, provide another approach to analyzing changing rates (cf. Coale, 1972, Ch. 6; Caswell, 1989, Ch. 8; Tuljapurkar, 1985). An explicit solution can be given for a sequence of 2 rate schedules in a population with 2 reproductive age groups (Schoen and Kim, 1994), but most cyclical populations are far too complex for direct algebraic solution.

The general case of a population whose vital rates are free to

change over time is even more complex. Coale (1972, Ch. 7) used Fourier analysis to link fertility schedules and their resulting birth sequences. While a valuable advance, his solution was quite complex and led him to conclude (p208), "the general case will always remain merely calculable, but not readily understandable at an intuitive level". Kim (1987), through an analysis of the discrete product matrix, found a general algebraic solution that connected changing rates and their birth sequences. Again, in most instances, her solution is too complex to render in closed form. Mitra (1990) examined some implications of a type of change in net maternity that led to a soluble form of Lotka's renewal equation.

Another related line of research focuses on "inverse projection" and "back projection" (cf. Lee, 1974, 1985; Oeppen, 1993; Wachter, 1986). Generally applied to studies in historical demography, inverse and back projection can be used to estimate a population's past age composition and levels of fertility and mortality from data on the total number of births and deaths. That approach shares the current interest in the long term relationship between births and fertility levels. However, it typically assumes a fixed age pattern of fertility and mortality and seeks numerical results that fit a given set of data, rather than setting forth a closed form model and examining relationships within that model.

Nonlinear models, where the vital rates receive feedback from the birth sequence, have been another area of research involving dynamic rates. Samuelson (1976) found an explicit solution for one particularly simple case, but analyses of nonlinear models typi-

cally encounter great complexity and can lead to "chaotic" behavior (cf. May, 1974; Wachter, 1991). A recent review of the area noted the continuing difficulty in relating birth sequences in nonlinear models to their underlying mortality and fertility behavior (Wachter, 1993).

SPECIFYING THE CONTINUOUS HYPERSTABLE MODEL

Given the formidable obstacles to finding the birth sequence generated by arbitrarily changing vital rates, our approach is to start with a specified birth sequence and look for consistent series of vital rates. We begin with Lotka's classic renewal formulation, rewritten to allow the vital rates to change over time. When the population initiating the renewal process does not contribute to births at time t, we can write

$$B(t) = \int_0^\infty B(t-x)p(x,t)m(x,t)dx \qquad (2)$$

where $p(x,t)$ is the cohort probability of surviving to age x at time t from birth at time t-x. For convenience, we introduce the net maternity function $\phi(x,t)$, where $\phi(x,t)=p(x,t)m(x,t)$, and the birth function $g(t)$, where

$$B(t) = B(0)g(t) \qquad (3)$$

In demographic applications, $g(t)$ should be nonnegative and we must have $g(0)=1$. Using equation (3), we can rewrite equation (2) as

$$1 = \int [g(t-x)/g(t)]\ \phi(x,t)\ dx \qquad (4)$$

Each age's proportional contribution to births at time t, $f(x,t)$,

is thus

$$f(x,t) = [g(t-x)/g(t)]\ \phi(x,t) \qquad (5)$$

and net maternity at age x and time t can thus be expressed in terms of the birth sequence and f(x,t). There are an infinite number of ways to do so, but we focus on three of them.

Assuming a fixed proportional distribution of births (f)

If each age's proportional contribution to the total number of births does not vary over time, i.e. if

$$f(x,t) = f(x) \qquad (6)$$

then equation (5) can be rewritten

$$\phi(x,t) = f(x)\ [g(t)/g(t-x)] \qquad (7)$$

A constant proportional distribution of births by age of mother is a demographically plausible assumption implicit in all stable populations. Under that "fixed f" assumption, equation (7) provides an age-time array of net maternity values consistent with g(t). The result is that equations (4) and (7) specify a hyper-stable model which replaces the stable population requirement of constant mortality and fertility with the much looser requirement of a constant proportional distribution of births by age of mother.

Assuming an Unchanging Age Pattern of Net Maternity (ϕ)

A second plausible assumption is that the age pattern of net maternity does not change, though its level varies over time. Indeed, proportional change at all ages is frequently assumed

because of the regularity of age patterns of net maternity. Mathematically, we can write

$$\phi(x,t) = k(t)\ \phi(x,s) \qquad (8)$$

where k(t) adjusts a base (or standard) net maternity $\phi(x,s)$ to yield the requisite birth sequence. Substituting equation (8) into equation (5) and integrating, using the fact that $\int f(x,t)=1$, yields

$$k(t) = 1 \Big/ \int [g(t-x)/g(t)]\ \phi(x,s)\ dx$$

and hence

$$\phi(x,t) = \phi(x,s) \Big/ \int [g(t-x)/g(t)]\ \phi(x,s)\ dx \qquad (9)$$

Equation (9) provides an array of net maternity rates consistent with a given birth sequence under the assumption that the proportional distribution of the ϕ rates is constant. Although the "proportional ϕ" assumption may seem simpler than the "fixed f" assumption, it requires integration and usually leads to a more complicated expression for net maternity.

Assuming a Constant Length of Generation (A)

A third approach is to apply the Mean Value Theorem to equation (4) and write

$$1 = [g(t-A(t))/g(t)] \int \phi(x,t)dx$$

where A(t) denotes the appropriate mean age at time t. If we assume the mean age is constant over time, we can write

$$R_o(t) = g(t)/g(t-A) \qquad (10)$$

where $R_o(t) = \int \phi(x,t)dx$ represents the NRR at time t. Mean age A is the length of time required for births to grow by a factor equal to their NRR, and thus can be viewed as a measure of generation length. Equation (10) provides a simple analytical relationship between the birth sequence and the level of net maternity. It does not specify a schedule of net maternity rates, but the Mean Value Theorem insures that such a schedule must exist[2].

Let us now define $\rho(t)$, the hyperstable growth rate of births at time t, by

$$\rho(t) = (d/dt) \ln g(t) \tag{11}$$

When the population initiating the renewal process does not contribute to births at time 0, we can use the fact that g(0)=1 to rewrite equation (11) as

$$g(t) = \exp\left[\int_0^t \rho(y)dy \right] \tag{12}$$

From equation (10), it follows that

$$R_o(t) = \exp\left[\int_{t-A}^t \rho(y)dy \right] \tag{13}$$

In general, equation (12) shows how the birth sequence is generated by $\rho(t)$. Under the "constant A" assumption, equation (13) shows that the NRR sequence is also readily generated by the hyperstable growth rate of births.

HYPERSTABILITY WITH SELECTED FORMS OF g(t)

The birth sequence g(t) can take many forms. Let us consider some specific examples and their associated hyperstable net mater-

nity functions under the fixed f assumption.

a. The Basic Exponential (Stable) Form

Let us begin with the case where births are growing exponentially, i.e. where

$$g(t) = \exp[h_1 t] \tag{14}$$

Then, from equation (7) and the continuing assumption that the population initiating the renewal process does not contribute to births at time 0, net maternity is given by

$$\phi(x,t) = \phi(x,0) \tag{15}$$

and, from equation (4), those constant rates yield the conventional stable population renewal (or characteristic) equation

$$1 = \int \exp[-h_1 x]\ \phi(x,0)dx \tag{16}$$

where h_1 is Lotka's r, the intrinsic rate of natural increase. Stability is thus a special case of hyperstability.

b. The Quadratic Exponential Form

Let the birth function be

$$g(t) = \exp[h_1 t + h_2 t^2] \tag{17}$$

Then the net maternity function is

$$\phi(x,t) = \phi(x,0)\ \exp[2h_2 xt] \tag{18}$$

and the hyperstable characteristic equation is

$$1 = \int \exp[-h_1 x + h_2 x^2]\ \phi(x,0)dx \tag{19}$$

Equations (17)-(19) show that when fertility increases exponentially over both age and time, the birth sequence is an exponentiated quadratic. Unlike the e^{kt} pattern of change used by Coale (1972) and shown in equation (1), the e^{kxt} pattern of equation (18) is associated with a simple closed form expression for the birth sequence.

The quadratic hyperstable model has potential applications to many analyses where the level of net maternity can be thought of as monotonically increasing or decreasing over time. That usefulness is enhanced because equation (18) describes a substantively meaningful pattern of fertility change. A decrease in net maternity over time (i.e. $h_2<0$) implies a greater decline in net maternity at higher ages relative to lower ages. That is consistent with the typical pattern of fertility decline and with the greater concentration of higher order births at older ages.

The parameters of the hyperstable model must satisfy only one constraint, that imposed by the characteristic equation. Thus when the time 0 net maternity schedule and h_2, its rate of exponential change, are known, h_1 can be found from equation (19). The value of h_1 is thus contingent on the value of h_2; h_1 is not the intrinsic growth rate implied by the $\phi(x,0)$ unless $h_2=0$.

Because both the birth sequence and the net maternity rates change in relatively simple ways, the structure of the quadratic hyperstable model can be explicated in some depth. However, that task is beyond the scope of the present paper.

c. The Cubic Exponential Form

When the birth function is

$$g(t) = \exp[h_1 t + h_2 t^2 + h_3 t^3] \tag{20}$$

the net maternity function is

$$\phi(x,t) = \phi(x,0)\ \exp[2h_2 xt + 3h_3 xt(t-x)] \tag{21}$$

and the hyperstable characteristic equation is

$$1 = \int \exp[-h_1 x + h_2 x^2 - h_3 x^3]\ \phi(x,0)dx \tag{22}$$

The cubic exponential form can model birth sequences that exhibit 2 changes in direction, and thus has the potential for a wide range of applications. The pattern of net maternity change indicated by equation (21) is not easy to visualize, but we find that it yields realistic age patterns over time. For example, Figure 1A shows $\phi(x,t)$ curves for 4 time points based on the net maternity pattern observed[3] in the United States, 1959-61. The age curve of net maternity exhibits a reasonable pattern at every time point, showing a slight skew to the right that increases as net maternity declines.

d. The General Exponentiated Polynomial Form

Let us now consider birth functions of the general form

$$g(t) = \exp[\Sigma_{j=1}^{n} h_j t^j] \tag{23}$$

where n is the order of the polynomial chosen. The associated net maternity function is

$$\phi(x,t) = \phi(x,0)\ \exp[\Sigma_{j=1}^{n} h_j \{t^j + (-x)^j - (t-x)^j\}] \tag{24}$$

and the hyperstable characteristic equation is

$$1 = \int \exp[\Sigma_{j=1}^{n} h_j(-x)^j] \, \phi(x,0)dx \qquad (25)$$

With an arbitrary number of parameters, this functional form can model virtually any birth sequence.

e. The Sinusoidal Form

Hyperstable populations can take on a number of cyclically repeating forms. Let us consider the birth function

$$g(t) = 1 + b \sin(2\pi t/T) \qquad (26)$$

where b is the amplitude of the sine wave ($|b| < 1$), and T is the period of cyclicity. The net maternity function is then given by

$$\phi(x,t) = \phi(x,0)[1+b\sin(2\pi t/T)][1-b\sin(2\pi x/T)]/[1+b\sin(2\pi\{t-x\}/T)] \qquad (27)$$

and the hyperstable characteristic equation is

$$1 = \int [1-b \sin(2\pi x/T)] \, \phi(x,0)dx \qquad (28)$$

Figure 1B shows net maternity functions at 4 time points for a sinusoidal hyperstable population with b=.2 and T=60 years. Despite the complexity of equation (27), those $\phi(x,t)$ curves show a conventional pattern and change in a reasonable manner. (The unimodal pattern can break down at very short cycle lengths, however. Marc Artzrouni has pointed out that with b=.32 and T=9.5 years, the ϕ curve becomes trimodal.)

f. The Double Exponential Form

If the rate of change in births is exponential, i.e. if

$$\rho(t) = ce^{-bt} \tag{29}$$

then

$$g(t) = \exp[\ (c/b)(1 - e^{-bt})\ ] \tag{30}$$

the net maternity function is given by

$$\phi(x,t) = \phi(x,0)\ \exp[\ (c/b)(1 - e^{-bt})(1 - e^{bx})\ ] \tag{31}$$

and the hyperstable characteristic equation is

$$1 = \int \exp[\ (c/b)(1 - e^{bx})\ ]\ \phi(x,0)dx \tag{32}$$

As shown in a later section, this double exponential form can model a population's transition to zero growth.

THE DISCRETE HYPERSTABLE MODEL

Specifying the Discrete Model

Under the fixed f assumption, the hyperstable model can be written in discrete time using a conventional population projection matrix (PPM) approach. The discrete form facilitates applications to data and provides the basis for analyzing convergence to hyperstability. The following discussion builds on the treatments of PPMs and convergence in Keyfitz (1977), Kim (1987), Caswell (1989), and Kim and Schoen (1993).

At time t, let $\mathbf{A}_t$ be the (n x n) PPM of a hyperstable population. The jth element of the first row of $\mathbf{A}_t$ is b_{tj}, the "fertility" rate applied to the number of persons in the jth age group at time t-1 to yield that group's contribution to the first age group at time t. The jth subdiagonal element of $\mathbf{A}_t$ is s_{tj}, the probability of surviving from age group j at time t-1 to age group

j+1 at time t. Let $\mathbf{x}_t$ be a vector whose jth element, x_{tj}, is the number of persons in the jth age group of the hyperstable population at time t. We then have the hyperstable population projection relationship

$$\mathbf{x}_t = \mathbf{A}_t\mathbf{x}_{t-1} \tag{33}$$

To make use of the fixed f assumption, let f_j be the fixed proportion of births contributed by persons in the jth age group. Then

$$\Sigma_j f_j = 1 \tag{34}$$

and at any time t we can write

$$f_j = b_{tj}x_{t-1,j}/x_{t1} \tag{35}$$

Using equation (35), we can write the nonzero elements of PPM $\mathbf{A}_t$ as

$$\mathbf{A}_t = \begin{bmatrix} f_1x_{t1}/x_{t-1,1} & f_2x_{t1}/x_{t-1,2} & \cdots & f_nx_{t1}/x_{t-1,n} \\ s_{t1} & & & \\ & s_{t2} & & \\ & & \ddots & \\ & & & s_{t,n-1} \end{bmatrix} \tag{36}$$

Now define $\mathbf{F}$ to be an (n x n) matrix which has f_j as the jth element in the first row, ones in the subdiagonal, and zeros elsewhere. Then $\mathbf{F}$ has the form of a PPM, and because of equation (34) its maximal eigenvalue is 1. With $\mathbf{X}_t$ an (n x n) diagonal matrix whose jth diagonal element is x_{tj}, we can write PPM $\mathbf{A}_t$ in the form

$$\mathbf{A}_t = \mathbf{X}_t\,\mathbf{F}\,\mathbf{X}_{t-1}^{-1} \tag{37}$$

as $s_{t,j-1} = x_{tj}/x_{t-1,j-1}$. The form of equation (37) is reminiscent of

a similarity transformation, but the subscripts of **X** are not identical. Equation (37) is an important relationship because it shows that any fixed f PPM can be decomposed into time dependent factors reflecting the hyperstable age distribution and a time invariant matrix reflecting the f_j.

With **M**(t,t+w) denoting the product matrix $\mathbf{A}_{t+w}\mathbf{A}_{t+w-1}...\mathbf{A}_{t+1}$, we have

$$\mathbf{x}_{t+w} = \mathbf{M}(t,t+w)\mathbf{x}_t \qquad (38)$$

and, from equation (37),

$$\mathbf{M}(t,t+w) = \mathbf{X}_{t+w}\,\mathbf{F}^w\,\mathbf{X}_t^{-1} \qquad (39)$$

Matrix **F**, which embodies the fixed proportional distribution of births, clearly plays a key role in hyperstable dynamics.

Convergence to Hyperstability

In general, we agree with Coale (1972, p119) that "If we can find an analytical expression for the birth sequence [that results from a specified net maternity pattern] we can be sure that it is indeed the birth sequence that would be approached ... because of the proven tendency for the effect of initial conditions to vanish with the passage of time." Under the fixed f assumption, we can go further and demonstrate mathematically how convergence to hyperstability occurs.

Assume that a specified population adopts a given regime of hyperstable vital rates beginning with PPM $\mathbf{A}_{a+1}$, and that $\mathbf{y}_t$ is the vector whose jth element, y_{tj}, is the number of persons in that population's jth age group at time t. We can then write

$$\mathbf{y}_{s+w} = \mathbf{A}_{s+w}\mathbf{A}_{s+w-1}\ldots\mathbf{A}_{s+1}\mathbf{y}_s = \mathbf{M}(s,s+w)\mathbf{y}_s \qquad (40)$$

The relative age distribution at time s+w is independent of the initial age distribution if $\mathbf{M}(s,s+w)$ is of rank 1 (Caswell, 1989, Ch. 8). When w is large, $\mathbf{M}(s,s+w)$ becomes a matrix of rank one by virtue of weak ergodicity. Population vector $\mathbf{y}_{s+w}$ has the hyperstable age distribution as the time s age distribution has been "forgotten".

Assume that matrix **F** is primitive, a conventional assumption that holds if any 2 elements in the first row, say f_i and f_j, are greater than zero and the greatest common divisor of $\underline{i}$ and $\underline{j}$ is 1. Matrix $\mathbf{F}^w$ must become a matrix of rank one when w is large because of strong ergodicity, the process that insures convergence to stability (cf. Caswell, 1989). Because the maximal eigenvalue of **F** is 1, $\mathbf{F}^w = \mathbf{u}_f \mathbf{v}_f'$, where $\mathbf{u}_f$ and $\mathbf{v}_f$ are the right and left eigenvectors of **F** normalized so that $\mathbf{v}_f'\mathbf{u}_f = 1$. Their $\underline{j}$th elements are $u_{fj} = 1$, and $v_{fj} = (\Sigma^n_{i=j} f_i)/\mu$, where the hyperstable mean age at childbearing $\mu = \Sigma_j jf_j$.

With $\mathbf{F}^w$ a matrix of rank one, the product matrix $\mathbf{M}(s,s+w)$ becomes, from equation (39),

$$\mathbf{M}(s,s+w) = (\mathbf{X}_{s+w}\mathbf{u}_f)(\mathbf{v}_f'\mathbf{X}_s^{-1}) \qquad (41)$$

also a matrix of rank one. Thus **M** becomes rank one ("converges") when $\mathbf{F}^w$ becomes rank one, and convergence to hyperstability parallels the convergence to stability of a population subject to unchanging PPM **F**. That is a significant finding because it not only shows that under the fixed f assumption the age composition of a population subject to a regime of hyperstable vital rates becomes

hyperstable, but indicates that exactly the same process brings about convergence in both stable and hyperstable models.

With $\mathbf{M}(s,s+w)$ a rank one matrix, we can rewrite equation (41) as a scalar times a product of vectors, i.e.

$$\mathbf{M}(s,s+w) = (x_{s+w,1}/x_{s1})\ \mathbf{u}_{s+w}\mathbf{v}_s' \qquad (42)$$

Vector $\mathbf{u}_{s+w}$, whose jth element is $u_{s+w,j} = x_{s+w,j}/x_{s+w,1}$, provides the relative hyperstable age distribution at time s+w. Vector $\mathbf{v}_s$, whose jth element is $v_{sj} = v_{fj}x_{s1}/x_{s-j+1,1}$, is a fixed vector of reproductive values. As shown by Kim and Sykes (1976), the age distribution depends only on recent rates while the reproductive values depend only on early rates. From equations (40) and (42), the hyperstable population at time s+w is completely specified by

$$\mathbf{y}_{s+w} = Q_s(x_{s+w,1}/x_{s1})\ \mathbf{u}_{s+w} \qquad (43)$$

where $Q_s = (\mathbf{v}'_s\ \mathbf{y}_s)$ is the total population reproductive value at initial time s.

Forward and Backward Growth Rates

Kim (1987) noted that when vital rates change over time, a population's "forward" and "backward" growth rates generally differ. Unlike the case where vital rates are constant, there are two characteristic equations that define two sets of changing growth rates. The forward growth rate at time t, λ_t, satisfies the forward characteristic equation (equation (25) of Kim, 1987)

$$\Sigma_j\ \phi_{tj}\ (\lambda_t\lambda_{t-1}\ldots\lambda_{t-j+1})^{-1} = 1 \qquad (44)$$

where $\phi_{tj} = b_{tj}p_{tj}$ is the net maternity rate of jth age group at time

t, with $p_{tj} = s_{t,j-1}s_{t-1,j-2}\cdots s_{t-j+1,1}$ representing the cohort survival probability from the first age group at time t-j to jth age group at time t. The backward growth rate at time t, γ_t, is determined by the backward characteristic equation (equation (44) of Kim, 1987),

$$\Sigma_j \ \phi_{t+j-1,j} \ (\gamma_t\gamma_{t+1}\cdots\gamma_{t+j-1})^{-1} = 1 \qquad (45)$$

In general, λ_t, the forward growth rate at time t in equation (44), is the ratio of the number of births (persons in the first age group) at time t to that at time t-1, while γ_t, the backward growth rate at time t in equation (45) is the ratio of population size in the distant future generated by one birth at time t-1 to that generated by one birth at time t.

In a hyperstable population that follows the fixed f assumption, equation (35) implies that $b_{tj} = f_j(x_{t1}/x_{t-1,j})$. Since $\phi_{tj} = b_{tj}p_{tj}$ and $x_{t-1,j}/p_{tj} = x_{t-j,1}$, equation (44) yields the hyperstable forward characteristic equation

$$\Sigma_j f_j \ (g_t/g_{t-j}) \ (\lambda_t\lambda_{t-1}\cdots\lambda_{t-j+1})^{-1} = 1 \qquad (46)$$

where $g_t = x_{t1}$. From equations (34) and (46), we confirm that

$$\lambda_t = g_t/g_{t-1} \qquad (47)$$

Similarly, the hyperstable backward characteristic equation is, from equation (45),

$$\Sigma_j f_j \ (g_{t+j-1}/g_t) \ (\gamma_t\gamma_{t+1}\cdots\gamma_{t+j-1})^{-1} = 1 \qquad (48)$$

and, from equations (34) and (48), we obtain

$$\gamma_t = g_t/g_{t-1} \qquad (49)$$

Thus under the fixed f assumption the forward and backward growth rates of a hyperstable population are identical, and the common growth rate at time t is given by g_t/g_{t-1}. Consequently, hyperstable population is a special case of a population with changing vital rates that requires the forward and backward growth rates to equal the growth rate of births.

ILLUSTRATIVE HYPERSTABLE CALCULATIONS

Patterns in 2 Hypothetical Hyperstable Populations

Let us examine the demographic behavior of some illustrative cubic and sinusoidal hyperstable populations. For the hypothetical cubic model whose age-specific net maternity rates are shown in Figure 1A, Figure 2A shows the g(t) function and the NRRs associated with our 3 alternative approaches for relating the g(t) and $\phi(x,t)$ functions. Over the 45 year time period shown, the birth function first rises gradually and then falls somewhat, while the reproductive level declines steadily. The 3 associated NRRs are virtually indistinguishable. Because cubic coefficient h_3 is positive, the population will eventually exhibit monotonically increasing birth levels and NRRs.

Figure 2B depicts the same demographic functions for the sinusoidal hyperstable population of Figure 1B. While g(t) is a true sine curve, the 3 NRR curves are only approximately sinusoidal. Relative to g(t), the NRR curves have a larger amplitude and a perceptible phase shift, even though their pattern remains the same from cycle to cycle. The fixed f and proportional ϕ approaches yield very similar NRRs, while the NRRs from the constant A

approach show somewhat greater variability at the extremes.

Figure 3A shows the age distributions for ages 0 through 50 for the cubic hyperstable population of Figures 1A and 2A, using mortality for U.S. females, 1960. Because of the nature of the hyperstable parameters, the number of births rises in the years before t=0. Thus the hyperstable population becomes older between t=0 and t=45, despite the relative flatness in the birth sequence during that interval. The age distribution appears to pivot at a point around age 23, although the exact pivot point increases slightly over time. Taken together, Figures 1A, 2A, and 3A depict the dynamics of a demographic change: as an initially growing population experiences a period of fertility decline, its birth sequence peaks and falls, and population aging proceeds.

Figure 3B presents age distributions for the sinusoidal hyperstable population of Figures 1B and 2B. The effects of the cyclical birth sequence are apparent in the waves that move through the population's age structure.

Actual and Hyperstable NRRs for the United States, 1960-89

The ability of NRRs produced by the fixed f, proportional ϕ, and constant A approaches to reproduce observed NRRs can be examined using the experience of the United States over the 1960-89 interval. The actual NRR values for 1960-89 were obtained[5] from Heuser (1976) and U.S. National Center for Health Statistics (1994). The births observed in the U.S. between 1920 and 1989 were taken from U.S. National Center for Health Statistics (1994, Table 1-1), and rescaled to show one birth in 1973. Using a base net

maternity pattern based on the U.S. 1970, NRRs were calculated under our 3 hyperstable assumptions. Figure 4 shows that the 3 hyperstable NRRs provide fairly good estimates of observed behavior, even though U.S. net maternity did not follow any of those patterns. While the robustness of hyperstable estimation has yet to be examined in depth, this example suggests that hyperstable estimates may be applicable to many observed populations.

AN APPLICATION TO MODELING TRAJECTORIES TO STATIONARITY

Recently there has been considerable interest in the ultimate size of the world's population, and how it would vary under different scenarios of fertility decline (e.g. Bos et al, 1992). Trajectories to stationarity under alternative assumptions about future declines in births can be calculated using the double exponential model of equations (29)-(32). In equation (30), the birth trajectory depends on 2 parameters. Parameter $\underline{b}$ represents the rate of decline in the number of births, while parameter $\underline{c}$ is related to the initial NRR. Under the constant A assumption, equations (10) and (30) yield

$$c = b \ln R_o(0) \; / \; (e^{bA} - 1) \tag{50}$$

The ultimate number of births can then be written

$$g(\infty) = R_o(0)^{[1/(e^{bA} - 1)]} \tag{51}$$

Not surprisingly, the size of the ultimate birth cohort is smaller when the initial NRR is lower, the rate of decline in births is greater, and the length of a generation is longer. However, the double exponential hyperstable model states the relationship

precisely, and makes it easier to compare the NRRs and the birth trajectories and ultimate levels that result from alternative assumptions.

Figure 5 shows the birth trajectories to stationarity for two populations with an initial NRR of 2.0 and A=27, but with different rates of decline in the number of births. For trajectory $g_1(t)$, $b_1 = .050$, and for trajectory $g_2(t)$, $b_2 = .025$. There is a very large difference in ultimate cohort size, with $g_1(\infty) = 1.27$ and $g_2(\infty) = 2.05$. While $NRR(\infty) = 1$ in both cases, net maternity drops much faster under the g_1 trajectory. Differences in NRRs based on the fixed f, proportional ϕ, and constant A assumptions are quite small.

SUMMARY AND CONCLUSION

Stable population models, based on fertility and mortality rates that do not change over time, are insufficiently flexible and realistic to capture the dynamics of populations with changing vital rates. Hyperstable models allow the rates to change, and relate a specified birth trajectory to a consistent set of net maternity schedules. Their potential usefulness in demographic modeling and estimation appears to be substantial, as they replace the static equilibrium of classical stability with the dynamic equilibrium of hyperstability.

FOOTNOTES

1. The focus here differs from the stochastic ergodicity of age distributions in populations with Markovian vital rates, a subject investigated by Cohen (1976, 1977).

2. In most cases, there appear to be an infinite number of solutions for $\phi(x,t)$, and it is usually not difficult to find one that is demographically realistic. However, when $g(t)$ is non-monotonic, the variance of $\phi(x,t)$ decreases when t approaches A years after an extremum. Exactly A years after an extremum of $g(t)$, all net maternity is concentrated at age A.

3. The mean and variance of net maternity in the United States 1959-61, as given in Keyfitz and Flieger (1968, p153), were used as the parameters of a Wicksell (gamma) function in order to produce the initial net maternity function (cf. Keyfitz, 1977, p156-57).

4. Total Fertility Rates from Heuser (1976, Table 1) were changed into NRRs by (i) multiplying by 100/205 to produce Gross Reproduction Rates (GRRs) and (ii) incorporating mortality by multiplying those GRRs by the probability of surviving to age 30. Survival probabilities were taken from period life tables appearing in Schoen (1987).

REFERENCES

Bos, Eduard, M.T. Vu, A. Levin, and R.A. Bulatao. 1992. World Population Projections. Baltimore: Johns Hopkins University Press.

Caswell, H. 1989. Matrix Population Models. Sunderland, MA: Sinauer.

Coale, A.J. 1972. The Growth and Structure of Human Populations. Princeton, NJ: Princeton University Press.

Cohen, J.E. 1976. Ergodicity of age structure in populations with Markovian vital rates. I: Countable states. Journal of the American Statistical Association 71:335-39.

Cohen, J.E. 1977. Ergodicity of age structure in populations with Markovian vital rates. II: General states. Advances in Applied Probability 9:18-37.

Heuser, R.L. 1976. Fertility Tables for Birth Cohorts by Color: United States, 1917-73. U.S. Dept Health Education and Welfare Publication No. (HRA) 76-1152, Rockville, MD.

Keyfitz, N. 1977. Introduction to the Mathematics of Population. Reading MA: Addison-Wesley.

Keyfitz, N. and W. Flieger. 1968. World Population: An Analysis of Vital Data. Chicago: University of Chicago Press.

Kim, Y.J. 1987. Dynamics of populations with changing rates: generalization of the stable population theory. Theoretical Population Biology 31:306-322.

Kim, Y.J. and R. Schoen. 1993. On the intrinsic force of convergence to stability. Mathematical Population Studies 4:89-102.

Kim, Y.J. and Z. Sykes. 1976. An experimental study of weak ergodicity in human populations. Theoretical population Biology 10:150-72.

Lee, R. 1974. Estimating series of vital rates and age structures from baptisms and burials: a new technique, with applications to pre-industrial England. Population Studies 28:495-512.

Lee, R.D. 1985. Inverse projection and back projection: a critical appraisal and comparative results for England, 1539 to 1871. Population Studies 39:233-48.

May, R.M. 1974. Biological populations with nonoverlapping generations: stable points, stable cycles, and chaos. Science 186:645-47.

Mitra, S. 1990. A birth model with oscillating rate of growth. Janasamkhya 8:35-40.

Oeppen, J. 1993. Back projection and inverse projection: members of a wider class of constrained projection models. Population Studies 47:245-67.

Samuelson, P. 1976. An economist's non-linear model of self-generated fertility waves. Population Studies 30:243-47.

Schoen, R. 1987. United States Marital Status Life Tables for Periods 1910-75 and Cohorts Born 1888-1945. Report No. NICHD/CPR/DBSB/87-1, NTIS Accession No. PB87-222485/AS.

Schoen, R. 1988. Modeling Multigroup populations. New York: Plenum.

Schoen, R. and Y.J. Kim. 1994. Cyclically stable populations. Mathematical Population Studies 4:283-95.

Tuljapurkar, S. 1985. Population dynamics in variable environments.

VI. Cyclical environments. Theoretical Population Biology 28:1-17.

United Nations. 1973. The Determinants and Consequences of Population Trends (Vol 1). New York: United Nations.

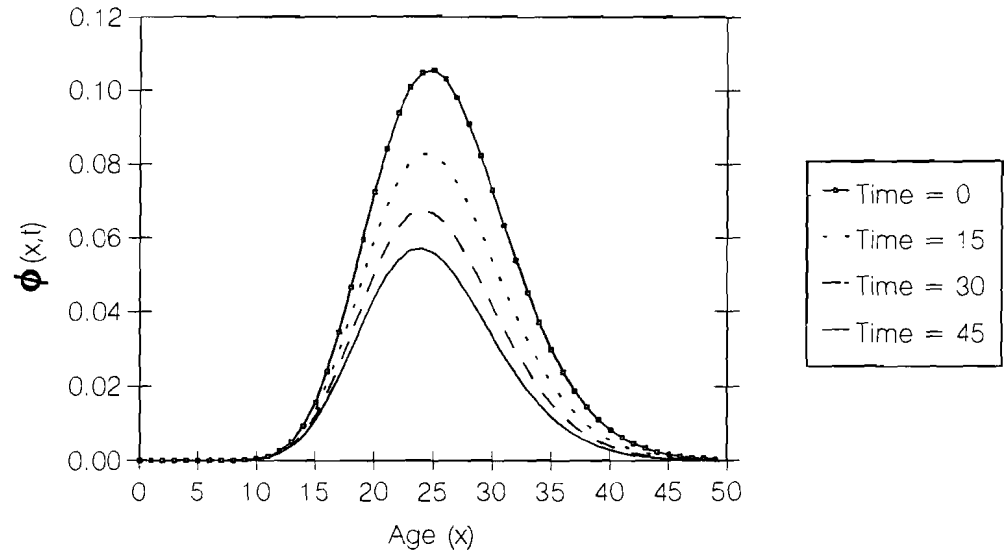
U. S. National Center for Health Statistics. 1994. Vital Statistics of the United States, 1990. Vol I, Natality. Washington DC: US Government Printing Office.

Wachter, K.W. 1986. Ergodicity and inverse projection. Population Studies 40:275-87.

Wachter, K.W. 1991. Elusive cycles: are there dynamically possible Lee-Easterlin models for U.S. births? Population Studies 45:109-35.

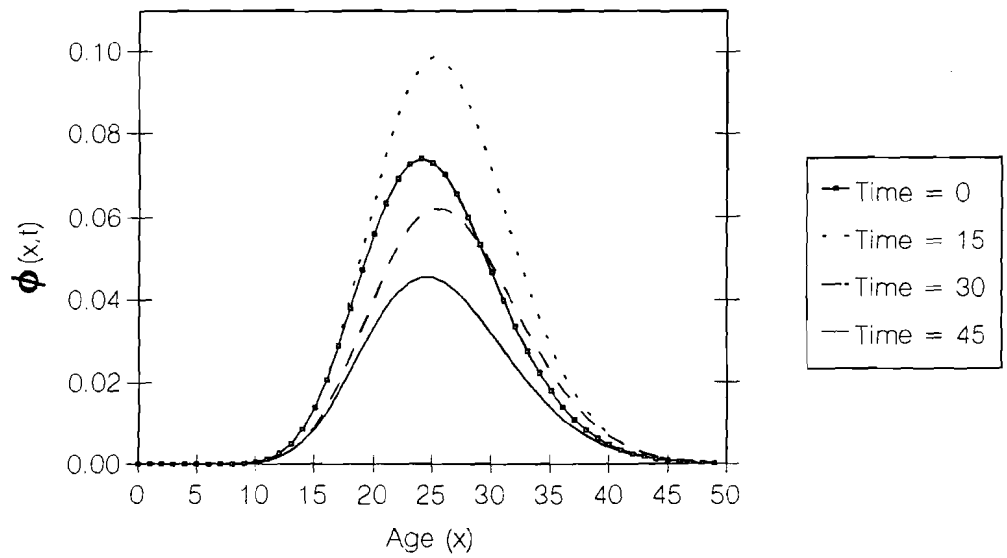
Wachter, K.W. 1993. The cohort feedback model with symmetric net maternity. Unpublished manuscript, Depts of Demography and Statistics, University of California, Berkeley.

Figure 1A. Cubic Hyperstable Age-Specific Fertility Curves, $\phi(x,t)$, at 4 Selected Time Points



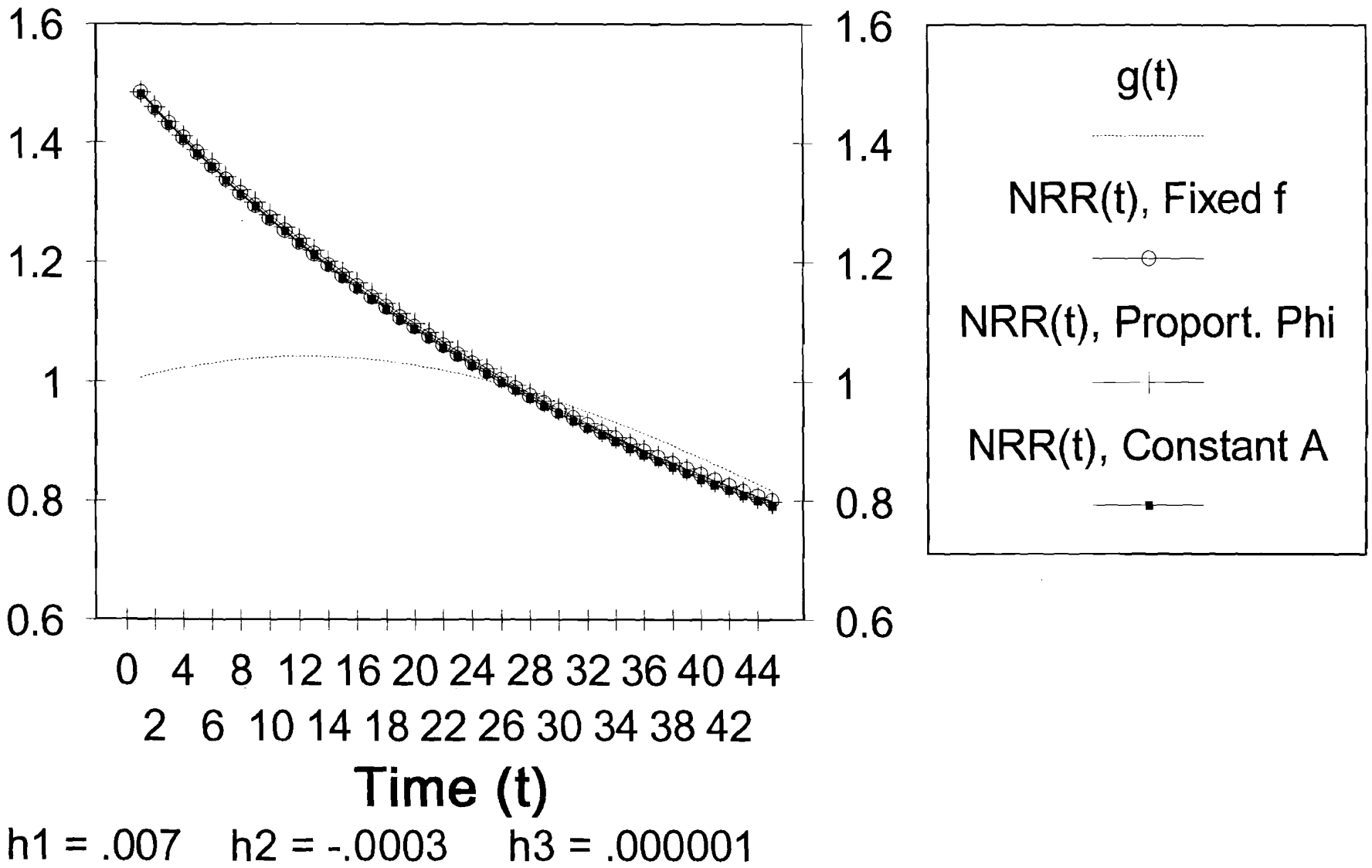
$h_2 = -.0003$ $h_3 = .000001$

Figure 1B. Sinusoidal Hyperstable Age-Specific Fertility Curves, $\phi(x,t)$, at 4 Selected Time Points



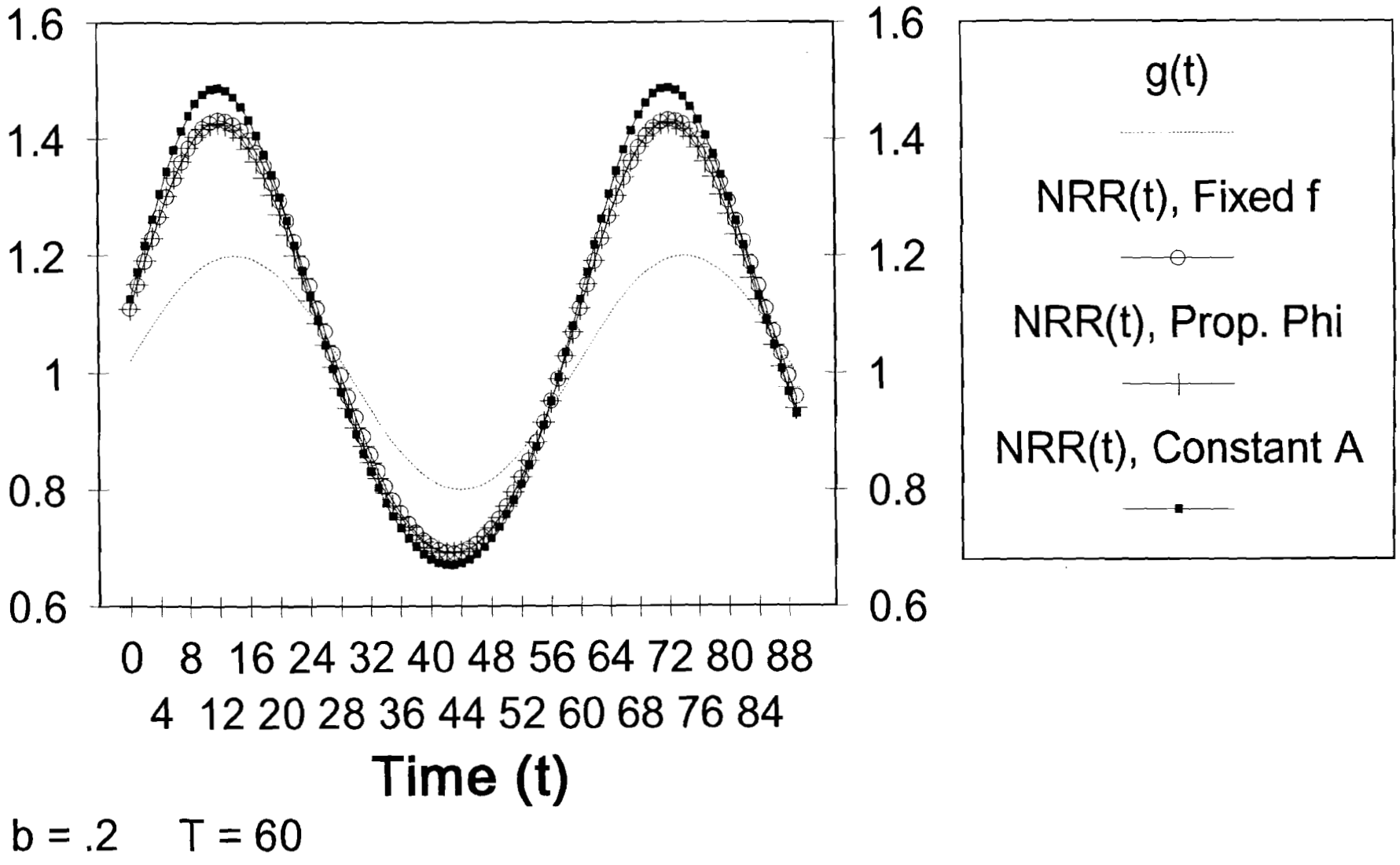
$b = .2$ $T = 60$

Figure 2A. Cubic Hyperstable Birth Function, g(t), and Associated Net Reproduction Rates, NRR(t).



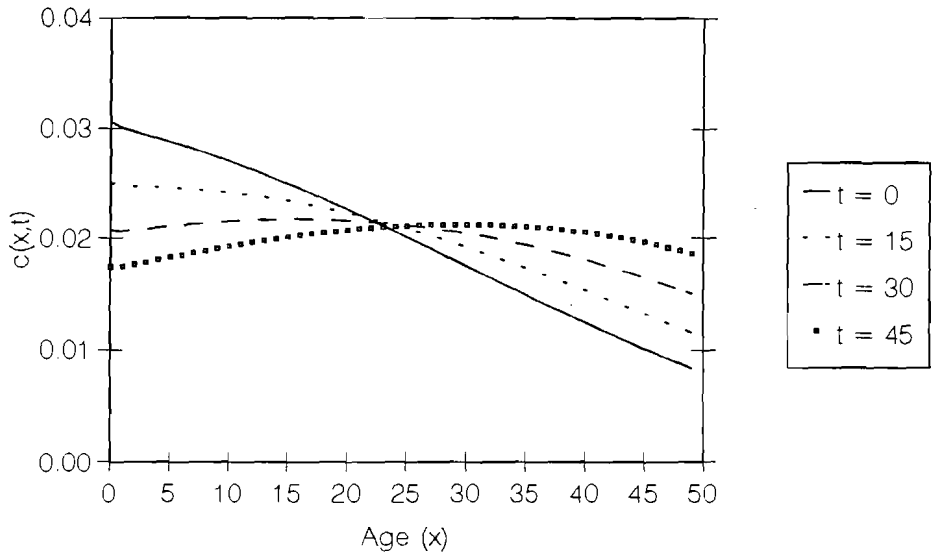
h1 = .007 h2 = -.0003 h3 = .000001

Figure 2B. Sinusoidal Birth Function, g(t), and Associated Net Reproduction Rates, NRR(t)



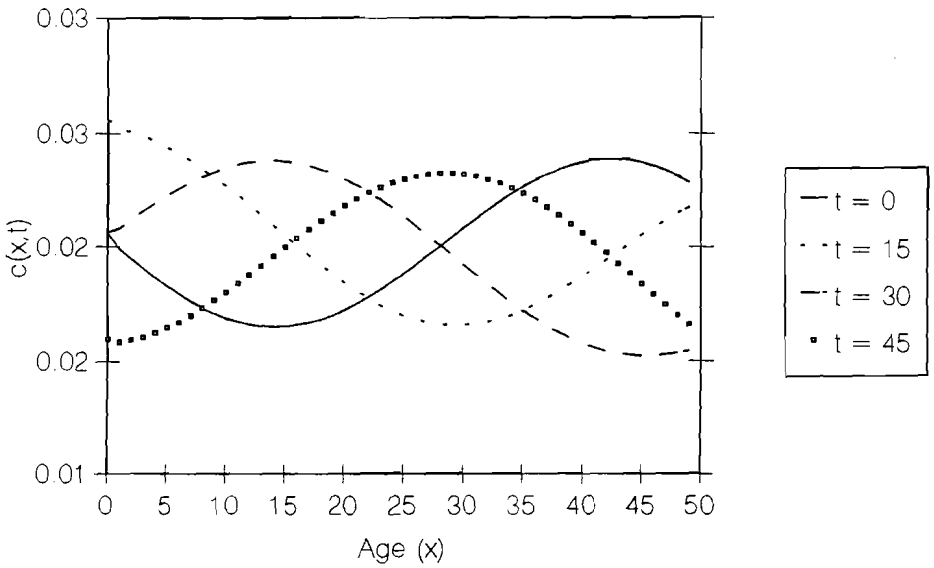
b = .2 T = 60

Figure 3A. Cubic Hyperstable Proportional Age Distributions, c(x,t), for Ages 0-50



$h_1 = .007$ $h_2 = -.0003$ $h_3 = .000001$

Figure 3B. Sinusoidal Hyperstable Proportional Age Distributions, c(x,t), for Ages 0-50



$b = .2$ $T = 60$

Figure 4. Observed and Estimated NRRs,

United States 1960-1989

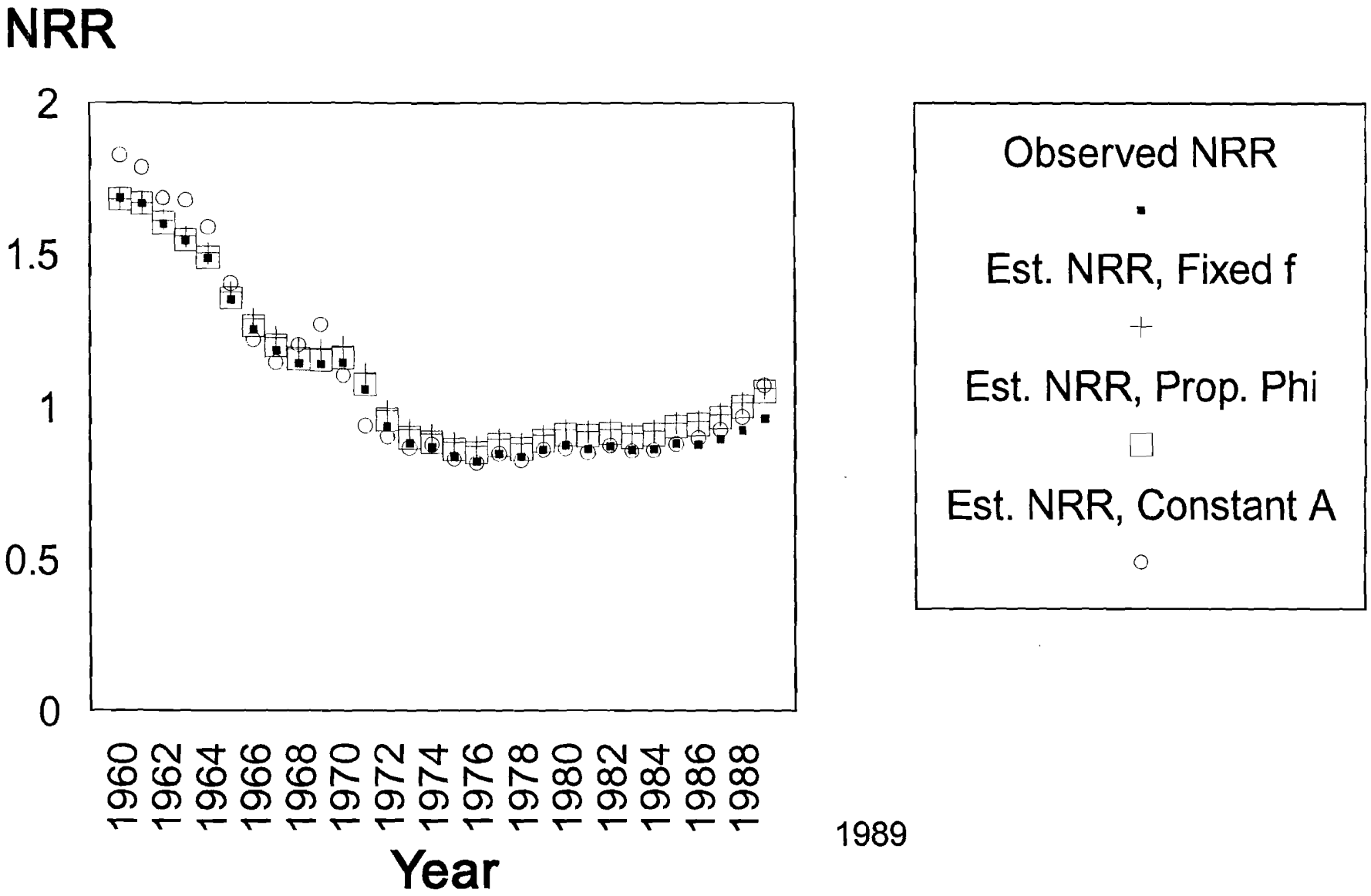
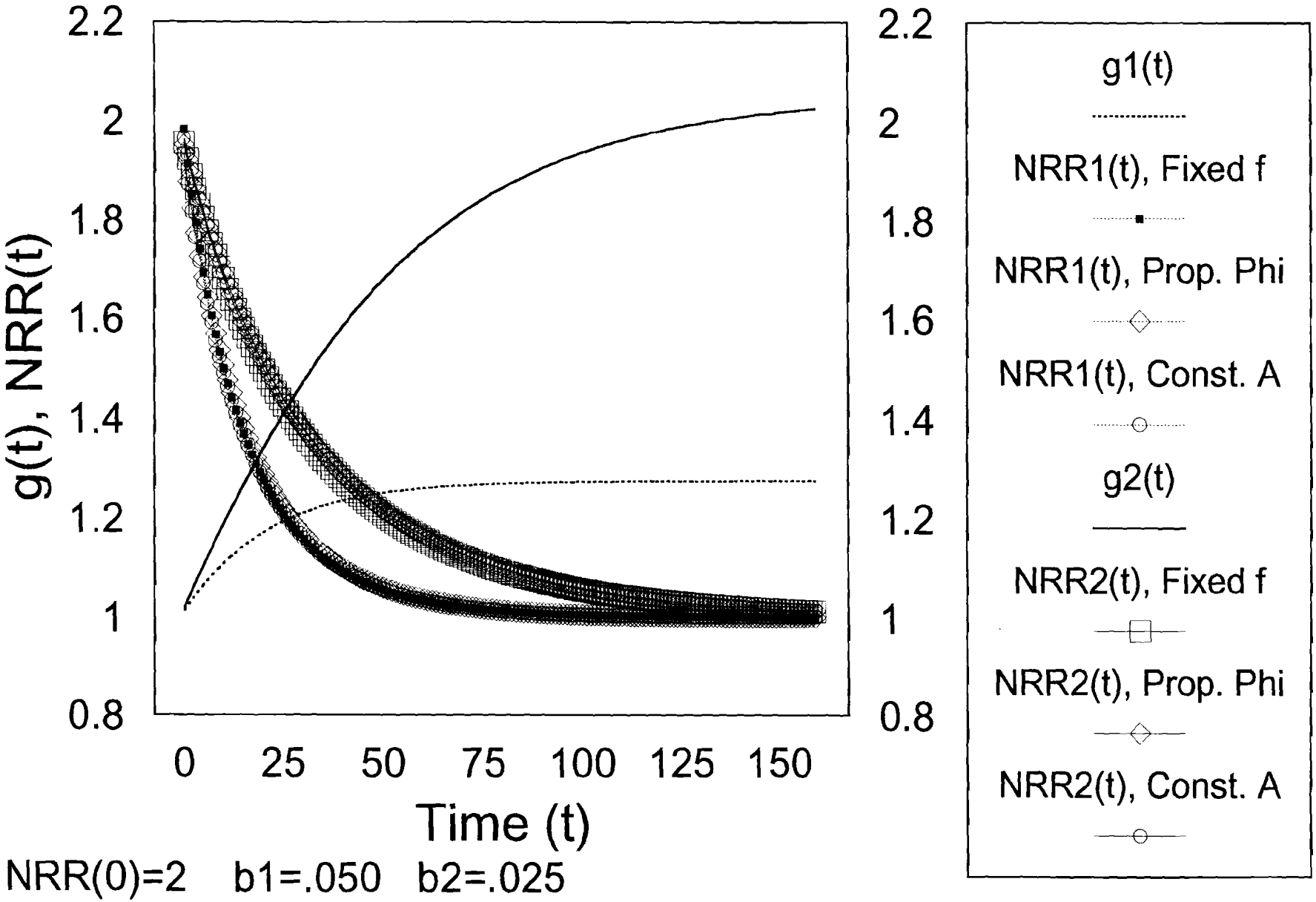


Figure 5. Double Exponential Birth Functions, g1(t) and g2(t) and Their Associated Net Reproduction Rates, NRR1(t) and NRR2(t).



NRR(0)=2 b1=.050 b2=.025

POPULATIONS WITH SUB-, SUPER-, AND INVERSE EXPONENTIAL GROWTH

Robert Schoen and Young J. Kim

Department of Population Dynamics
Johns Hopkins University
Baltimore MD 21205

This work was supported by grants R01 HD19145 and HD28443 from the Center for Population Research (NICHD), and benefitted from support provided to the Hopkins Population Center by NICHD grant P30 HD06268 and NCRR Shared Instrumentation Grant 1 S10 RR07268. Research assistance from Robin M. Weinick and Yan Yan is acknowledged with thanks.

Populations with Sub-, Super-, and Inverse Exponential Growth

ABSTRACT

The constant exponential growth of the classical stable population describes just one of an infinite number of possible birth trajectories. Here we examine birth functions of the form $\exp[ht^{\gamma}]$, which produce models with sub-, super-, and inverse exponential growth. Those "γ-stable" populations display changing vital rates and a growth parameter (h) whose properties are quite unlike stable growth rate r. The results expand the tools available to demographers, and provide new ways to model populations experiencing positive environmental feedback, density inhibited growth, and the transition to stationarity.

Populations with Sub-, Super-, and Inverse Exponential Growth

The central model of mathematical demography is the stable population, whose exponentially increasing births and unchanging age composition follow from constant age-specific birth and death rates (Lotka, 1939; Keyfitz, 1977). With g(t)dt the number of births between times t and t+dt, the stable birth trajectory is

$$g(t) = e^{rt} \qquad (1)$$

where g(0)=1 and parameter r is Lotka's "intrinsic" growth rate. The implicit growth model underlying stable populations is expressed by the differential equation

$$d \ln g(t)/dt = r \qquad (2)$$

which demonstrates that the growth rate of the birth function is constant over time.

The present paper goes beyond simple exponential growth by allowing the growth rate of births to change over time. It examines a broader set of birth functions, and explores how those sub-, super-, and inverse exponential models relate to stable populations with the same vital rates. A major objective is to go beyond fixed vital rates and broaden the range of plausible models capable of reflecting the dynamics of population change.

GENERATING ALTERNATIVE BIRTH TRAJECTORIES

There is nothing sacrosanct about assuming that the birth trajectory is exponential or that the growth rate is constant over

time. Growth typically involves many factors that change over time, and considerations of density dependence suggest that the absolute number of births also has an effect (cf. May, 1976).

In statistical analyses, Weibull functions are frequently used to provide more flexibility than that afforded by the simple exponential. With that in mind, let us consider birth functions of the form

$$g(t) = \exp[\, ht^{\gamma} \,] \tag{3}$$

We term these more general models "γ-stable", as they differ from stable models by the introduction of the "shape" parameter γ. At time t, the growth rate of births in a γ-stable population is given by

$$d \ \ln g(t)/dt = \gamma \ [\ln g(t)] \ / \ t$$

and the growth rate of log births by

$$d \ln \ln g(t)/dt = \gamma \ / \ t \tag{4}$$

Thus growth in γ-stable models is inversely related to time and reflects size effects through the nonlinear nature of the natural logarithm function. The function $\exp[\, ht^{\gamma} \,]$ generalizes the classical stable formulation, and those birth trajectories and their implications are the focus of this paper.

THE CHARACTERISTIC EQUATION OF γ-STABLE POPULATIONS

The characteristic (or renewal) equation of any population can be written

$$1 = \int [g(t-x)/g(t)] \ \phi(x,t) \ dx \tag{5}$$

where $\phi(x,t)dx$ represents the net maternity rate between ages x and x+dx at time t, and the integral ranges over all reproductive ages. With g(t) of the form shown in equation (3),

$$1 = \int \exp[\ h\{(t-x)^{\gamma} - t^{\gamma}\}\ ]\ \phi(x,t)\ dx \qquad (6)$$

Equation (6) is the characteristic equation of an γ-stable population. Here we assume that the γ-stable population came into existence at a time well before time 0, so we can consider times before ß (the maximum age at childbearing). Given γ, time t and the schedule of net maternity rates, equation (6) can be used to find γ-stable growth rate h, and that effort can illuminate some of the distinctive features of γ-stable models.

DETERMINING γ-STABLE GROWTH PARAMETER h

A basic element of stable population theory is that the stable population characteristic equation

$$1 = \int e^{-rx}\,\phi(x)\ dx \qquad (7)$$

has a unique real solution for intrinsic growth rate r, and that r is greater than the real part of any complex root satisfying equation (7) [cf. Keyfitz, 1977, p100; Pollard, 1973, p24-5]. The same cannot be said for γ-stable growth rate h.

To examine possible solutions for h, let us define the function

$$\Psi(h) = \int_0^{ß} \exp[\ h\{(t-x)^{\gamma} - t^{\gamma}\ }]\ \phi(x,t)dx \qquad (8)$$

whose derivative with respect to h is

$$d\Psi(h)/dh = -\int_0^{\beta} [t^{\gamma}-(t-x)^{\gamma}] \exp[\ h\{(t-x)^{\gamma} - t^{\gamma}\ \}]\ \phi(x,t)dx \qquad (9)$$

With $\gamma>0$ and $t\geq\beta$, $[t^{\gamma}-(t-x)^{\gamma}]>0$, and $d\Psi(h)/dh$ decreases monotonically. Equation (8) indicates that as $h\rightarrow-\infty$, $\Psi(h)\rightarrow\infty$, and as $h\rightarrow\infty$, $\Psi(h)\rightarrow0$. Since $\Psi(h)$ is a continuous function of h, and the functions $[t^{\gamma}-(t-x)^{\gamma}]$, $\exp[\ h\{(t-x)^{\gamma} - t^{\gamma}\ \}]$, and $\phi(x,t)$ are all non-negative over the interval of integration and all simultaneously positive over part of that interval, there must be one and only one point where $\Psi(h)=1$. In this case, which described in the first row of Table 1, the line of argument that has been used to analyze stable populations (e.g. Pollard, 1973, p25) applies to γ-stable populations.

When $0<t<\beta$, the argument for 1 real h holds only if $\gamma>0$ is a rational number that can be expressed as the ratio of odd integer p divided by odd integer q, where p and q share no common factors. In that case (Table 1, row 2, column 1), $[t^{\gamma}-(t-x)^{\gamma}]$ never becomes negative. In contrast, if γ is a rational number of the form odd p divided by even q, or is irrational, $(t-x)^{\gamma}$ becomes complex, and there is no real h (Table 1, row 2, column 3).

The most striking case is when γ is of the form even p divided by odd q (Table 1, row 2, column 2). When $\frac{1}{2}\beta<t\leq\beta$, $[t^{\gamma}-(t-x)^{\gamma}]$ never becomes negative and there is one real h. However, when $0<t<\frac{1}{2}\beta$, $[t^{\gamma}-(t-x)^{\gamma}]$ does become negative at some ages. The value of $\Psi(h)$ then approaches ∞, not 0, when $h\rightarrow\infty$, and there are 0, 1, or 2 real solutions for h depending on whether the unique minimum of $\Psi(h)$ is greater than, equal to, or less than 1. (That minimum is unique

because there is only one point where $d\Psi(h)/dh=0$.) We have examined this case numerically, finding that there are indeed values of γ, t, and demographically realistic ϕ that yield 0 or 2 real solutions for h. For example, when $\gamma=34/55$ and the net maternity rates are those observed for United States Females, 1964, there are no solutions for h when t=14 and two solutions when t=15.

Table 2 presents some numerical analyses of the nature of the 2 real solutions for h. The calculations begin with a population at time $t=\beta=50$ that has net maternity given by a Wicksell (Gamma) distribution with mean 27 years, variance 35 years, and a Net Reproduction Rate [R(50)] of 1.5. Given γ and $\phi(x,t)$, equation (6) can be used to find, by iteration, the unique real growth rate, h_o. By appropriately changing the ϕ values[1], we can preserve solution h_o as t varies between 0 and $\frac{1}{2}\beta$, and iteratively find the second real solution, h_1. Table 2 shows values of h_o and h_1 for $\gamma=2$ and $\gamma=2/3$. At t=1, with h_o positive and somewhat small, $h_1(1)$ is large and negative and R(1) is below replacement. As t increases to $\frac{1}{2}\beta$, $h_1(t)$ and R(t) increase, with h_1 rising sharply as t approaches 25. With t around 13-14 years, $h_1(t)$ becomes zero as R(t) becomes one. Inspection of equation (6) indicates that h=0 is always a solution when R(t)=1.

Another instance when the behavior of γ-stable populations departs from that of stable populations arises when $t<0$ and γ is either irrational or of the form odd p divided by even q (Table 1, row 3, column 3). Then there is no real solution for h, but there is a unique complex h that yields a demographically meaningful

population (i.e. one with real values for all births and real vital rates). To find that solution, let $\gamma>0$ and rewrite equation (6) as

$$1 = \int_0^{\beta} \exp[\, h(-1)^{\gamma}\{(t^*+x)^{\gamma} - (t^*)^{\gamma}\}\,]\ \phi(x,t)dx \qquad (10)$$

where $-t=t^*>0$. The expression $\{(t^*+x)^{\gamma} - (t^*)^{\gamma}\}$ is always real and positive. If $h=h^*(-1)^{-\gamma}$, with h^* real, then an examination of $\Psi(h^*)$ shows that equation (10) yields 1 real solution for h^*. A similar argument holds when $\gamma<0$. Although h itself is complex, the whole exponential term in equation (10) is always real, and the characteristic equation describes a model that is demographically valid.

More generally, when $\gamma<0$, Table 1b shows that the nature of possible solutions for h is similar to the situation when $\gamma>0$. The possible solutions shown follow from the same line of argument pursued above, i.e. analysis of the characteristic equation and the $\Psi(h)$ function. When $\gamma<0$, the major difference is that the birth trajectory is typically discontinuous at time 0, with $g(0)=\infty$. In one instance $g(0)=0$, but $g(0)$ never equals 1 as it does when $\gamma>0$.

In stable populations, real root r is always a maximal root, i.e. is always greater than the real part of any complex root. In γ-stable populations, Table 1 shows that a unique real h is not always a maximal root[2]. Following the birth trajectory implied by the unique real h leads to different complex roots at different points in time.

γ-STABLE PATTERNS OF BIRTH AND REPRODUCTION

For different values of γ and real h, γ-stable populations can exhibit the 12 qualitatively different birth trajectories shown in

Figure 1. Inverse exponential populations, those where $\gamma<0$, always approach 1 as $t\to\infty$ (and where defined, when $t\to-\infty$ as well). Sub-exponential populations (where $0<\gamma<1$) and super-exponential populations (where $\gamma>1$) always approach either 0 or ∞ at the extremes of time.

There is no unique way to relate birth trajectory g(t) to Net Reproduction Rate trajectory R(t). A convenient way is to apply the Mean Value Theorem to characteristic equation (6) and, assuming that mean age A does not change over time (Schoen and Kim, 1994), write

$$R(t) = g(t)/g(t-A) = \exp[\, h\{t^{\gamma} - (t-A)^{\gamma}\}] \qquad (11)$$

We use that relationship here because of its simplicity and because we do not focus on the age pattern of net reproduction.[3]

For $t>\beta$, Table 3 describes how g(t) and R(t) are related. Whenever $t>\beta$, births do not increase when $R(t)<1$ nor decrease when $R(t)>1$. In both inverse and sub-exponential populations, g(t) and R(t) move in opposite directions, with R(t) always moving toward 1. However, sub-exponential populations do not approach zero growth. Instead, they provide examples of density dampened growth, where population size can continue to increase even though reproductivity is steadily decreasing. In super-exponential populations, there is a positive response to increasing density, and both g(t) and R(t) move in the same direction.

When $0<t<\beta$, the form of γ must be considered. If γ is of the form odd p divided by even q (or is irrational), the net reproduction rates are not defined for real h. When $\gamma<0$, R(t) is also not

defined for $0<t<\beta$ because either $g(0)$ or $R(A)$ is infinite.

When $\gamma>0$ is of the form even p divided by odd q, $g(t)$ changes direction at $t=0$. $R(t)$ moves in a more complex fashion, though it increases (or decreases) monotonically outside of the $0<t<\beta$ interval (see Figure 2). Inspection of equation (11) shows that $R(A)=g(A)$, and that $R(\frac{1}{2}A)=1$.

When $\gamma>0$ is odd p/odd q, $g(t)$ changes monotonically, and we again have $g(A)=R(A)$. In this case, $R(t)$ is never equal to 1, and has a global extremum at $t=\frac{1}{2}A$ (see Figure 2).

In reality, the form of γ may not be evident. Consider the Fibonacci series 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89 ..., where every term after the first two is the sum of the two preceding terms. It is well known that the ratio of successive terms approaches the limit $\frac{1}{2}(5^{\frac{1}{2}}-1)\approx.618034$. For example, to 6 digits 34/55=.618182 and 55/89=.617978. Hence a population exhibiting growth at $t>\beta$ consistent with $\gamma\approx.618$ could have a γ that is irrational, even p/odd q, or odd p/odd q. If that γ is irrational (or of the form odd p/even q), $R(t)$ is not defined below $t=\beta$. Figure 2 shows the results for the other 2 cases, with A=28 and h=.01. Panel a shows that the two $g(t)$ functions cannot be distinguished when $t\geq0$, but diverge markedly for $t<0$. Panel b shows that for both $\gamma=34/55$ and $\gamma=55/89$, $g(28)=R(28)=1.082$. Below time $\beta=50$, net reproductivity when $\gamma=34/55$ diverges from that of $\gamma=55/89$. For $\gamma=55/89$, $R(t)$ is always above 1, reaching a maximum of 1.108 when t=14. The $\gamma=34/55$ reproductive level at t=14 is 1, declines to a minimum of .925 at t=0, and remains below 1 for $t<0$.

In γ-stable populations, the same behavior at $t \geq \beta$ can be associated with three very different patterns at $t < \beta$.

γ-STABLE AND ASSOCIATED STABLE POPULATIONS

At every point in time, a γ-stable population has an associated stable population, i.e. the stable population determined by the $\phi(x,t)$ values then prevailing in the γ-stable population. Here we explore the relationship between the birth trajectories of those two populations.

For any γ-stable population, let us consider birth ratio $Z(y,t)$, the number of births in the γ-stable population y years before reference year t divided by the number of births in that same year in the time t associated stable population, assuming that both populations have the same number of births in year t. With $r(t)$ denoting the intrinsic growth rate of the time t associated stable population, we can write

$$Z(y,t) = [g(t-y)/g(t)] \; / \; e^{-r(t)y}$$
$$= \exp[\; h(t-y)^{\gamma} - ht^{\gamma} + r(t)y \;] \qquad (12)$$

Taking logs and expanding $(t-y)^{\gamma}$ using a Taylor series

$$\ln Z(y,t) = r(t)y - ht^{\gamma} + h[\; t^{\gamma} + (-y)\gamma t^{\gamma-1} + \tfrac{1}{2}(-y)^2\gamma(\gamma-1)t^{\gamma-2}$$
$$+ (1/3!)(-y)^3\gamma(\gamma-1)(\gamma-2)t^{\gamma-3} + \ldots \;] \qquad (13)$$

The characteristic equation of the associated stable population, under the assumption of a fixed proportional distribution of births by age of mother, can be written (Schoen and Kim, 1994)

$$1 = \int e^{-r(t)x}[g(t)/g(t-x)]\, f(x)\, dx \qquad (14)$$

where $f(x)$, with $\int f(x)=1$, is the fixed renewal density. From the Mean Value Theorem, there must be some value $A^*(t)$ for which we can rewrite equation (14) as

$$1 = [g(t-A^*(t))/g(t)]/e^{-r(t)A^*(t)} = Z(A^*(t),t) \qquad (15)$$

Again taking logs and using a Taylor series expansion of $(t-A^*(t))^{\gamma}$,

$$r(t) = h\gamma t^{\gamma-1}[\ 1 - \tfrac{1}{2}A^*(t)(\gamma-1)t^{-1} + (1/3!)[A^*(t)]^2(\gamma-1)(\gamma-2)t^{-2} - \ldots\ ] \qquad (16)$$

Substituting equation (16) into equation (13)

$$\ln Z(y,t) = \tfrac{1}{2}yh[y-A^*(t)]\gamma(\gamma-1)t^{\gamma-2} - (1/3!)yh[y^2-\{A^*(t)\}^2]\gamma(\gamma-1)(\gamma-2)t^{\gamma-3} + \ldots \qquad (17)$$

Equation (17) illuminates the relationship between the γ-stable and the time t associated stable birth trajectories. The γ-stable and time t associated stable populations always have the same number of births in reference year t and in year $t-A^*(t)$. In other years, the relative number of births depends on γ and t. The first and largest term on the right side of equation (17) varies with $t^{\gamma-2}$. If $\gamma>2$, the γ-stable and time t associated stable birth trajectories diverge over time at all x values other than 0 and $A^*(t)$. If $\gamma<2$, however, the γ-stable and the time t associated stable populations approach the same birth trajectory. In effect, when $\gamma<2$, the time t γ-stable population eventually becomes indistinguishable from the time t associated stable population.[4]

SUMMARY AND CONCLUSIONS

Sub-, super-, and inverse exponential growth models generalize

stable population theory, provide for changing vital rates, and reflect feedback from population size. While the specification of γ-stable birth trajectories in equation (3) is quite simple, the models display substantial flexibility and surprising complexity.

The uniqueness of γ-stable growth parameter h depends on both γ and time (t). Even though the model is assumed to come into existence when time has a large negative value, complexities arise at times less than the maximum age at childbearing (β). As shown in Table 1, under different circumstances a given γ and set of vital rates can imply 0, 1, or 2 real solutions for h. In one instance, a demographically meaningful population model can be specified with a unique but complex growth rate h.

Figure 1 shows the 12 qualitatively different birth trajectories that can arise in γ-stable populations. Depending on γ, net reproduction either attains replacement level or experiences an extremum between times 0 and β. Table 2 describes patterns of births and net reproduction when $t>\beta$. As exponentially growing populations can experience growth with constant levels of reproduction, sub-exponential growth populations (where $0<\gamma<1$) can exhibit a continually increasing birth trajectory with a monotonically decreasing level of reproduction.

At any point in time, the birth trajectory of an γ-stable model can be compared with that of the associated stable population that has the same vital rates. When $\gamma>2$, the birth trajectories of the γ-stable and associated stable populations diverge for large values of t. However, when $\gamma<2$, the time t γ-stable and the time

t associated stable birth trajectories converge for very large t.

Models with sub-, super-, and inverse exponential growth provide a new perspective on population growth and the implications of density dependence. They can provide a useful framework for modeling many areas of demographic interest, including the approach to zero growth, positive returns to scale, and negative feedback.

FOOTNOTES

1. Schoen and Kim (1994) examined the case where the renewal density (i.e. the proportional distribution of births by age of mother) does not change over time, and h is a constant. Under that "fixed f" assumption

$$\phi(x,t) = \phi(x,50) \{ g(t)g(50-x)/[g(50)g(t-x)] \}$$

and that relationship was used in calculating Table 2. Schoen and Kim (1994) also proved that under the fixed f assumption, any initial population converges to hyperstability. No convergence proof is available for the constant A assumption. However, consistent with the principle of weak ergodicity, convergence has never failed to occur in our numerical work.

2. The proof that h is a non-maximal root in certain circumstances follows from the same argument used in the stable case (e.g. that in Pollard, 1973, p25). For example, when $t \leq 0$ and $\gamma > 0$ is of the form even p/odd q, one finds the same form as Pollard's equation (3.2.9), but with a difference in sign indicating that the real part of the complex root is greater than the real root. Using several values of γ, we have verified that inequality numerically.

3. In γ-stable models, we have found that the constant A and fixed f assumptions lead to very similar numerical results.

4. When $\gamma=2$, the hyperstable and associated stable populations neither converge nor diverge. Using the Kullback distance to relate the time t hyperstable population to its associated stable population, Kim and Schoen (1996) show that the two populations remain the same distance apart for all values of t.

REFERENCES

Keyfitz, N. 1977. Introduction to the Mathematics of Population (2d ed.) Reading, MA: Addison-Wesley.

Kim, Y.J. and Schoen, R. 1996. Populations with quadratic exponential growth. Mathematical Population Studies, forthcoming.

Lotka, A.J. 1939. Theorie analytique des associations biologiques. Paris: Hermann.

May, R.M. 1976. Theoretical Ecology: Principles and Applications. Philadelphia: Saunders.

Pollard, J.H. 1973. Mathematical Models for the Growth of Human Populations. Cambridge: Cambridge Univ Press.

Schoen, R. and Kim, Y.J. 1994. Hyperstability. Paper presented at the Annual Meeting of the Population Association of America in Miami FL.

Figure 1. Birth trajectories by form and sign of gamma and sign of h

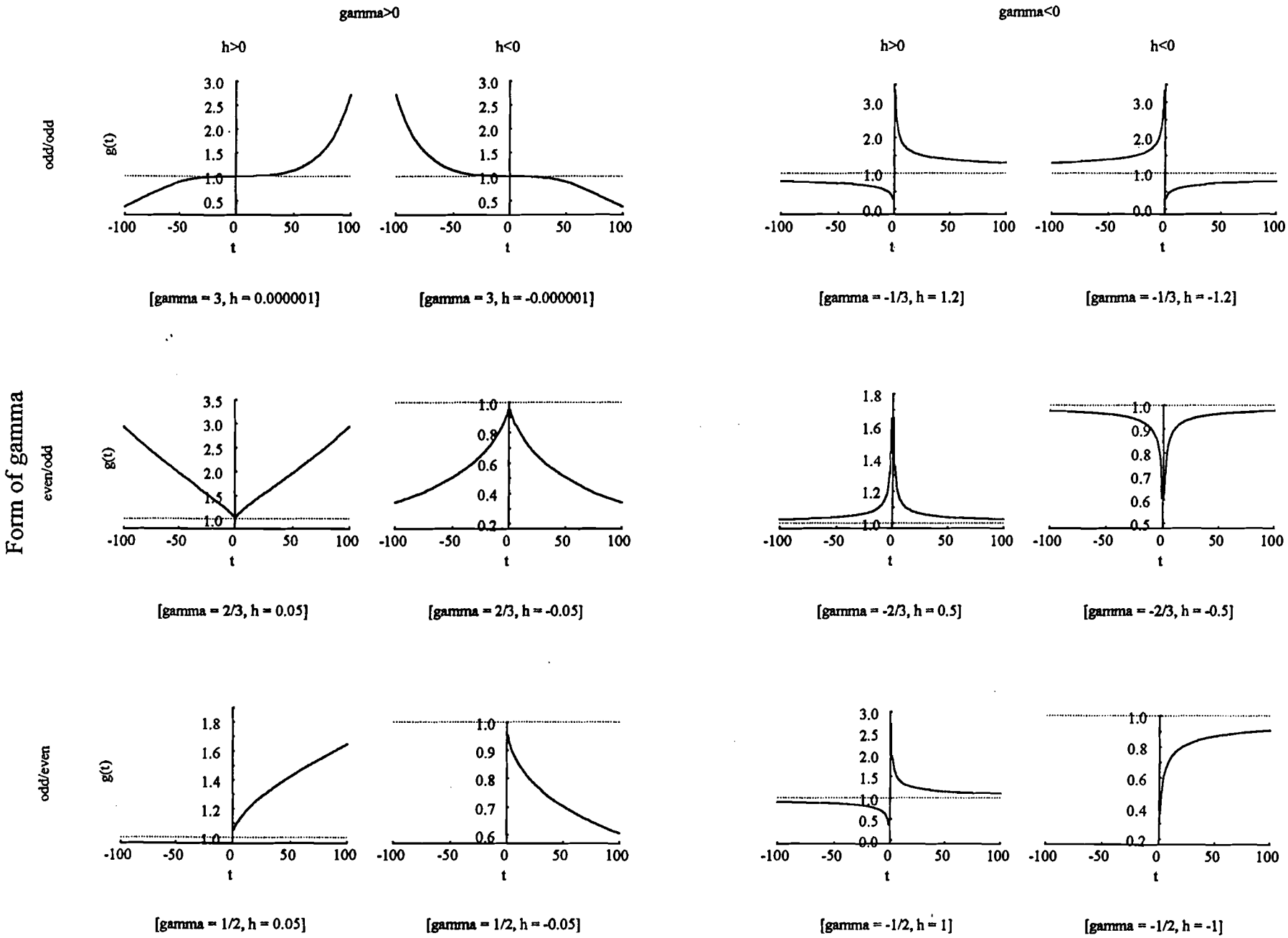


Figure 2. Birth (g) and Net Reproduction (R) functions when gamma (γ) is 34/55 and 55/89

Figure 2a. g(t) function

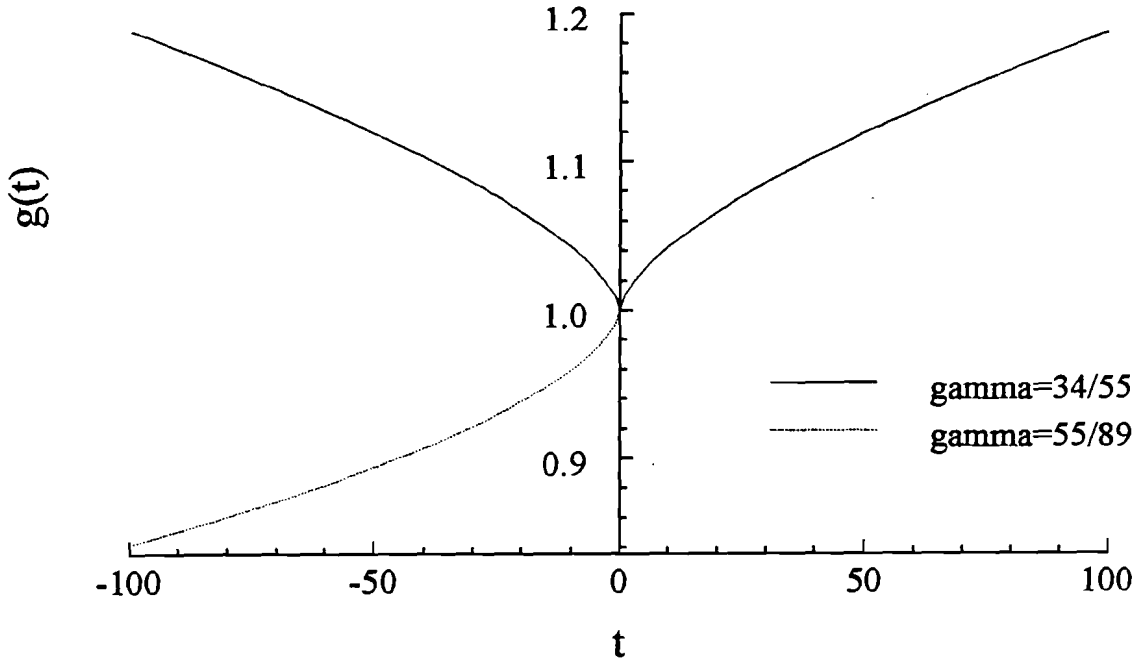
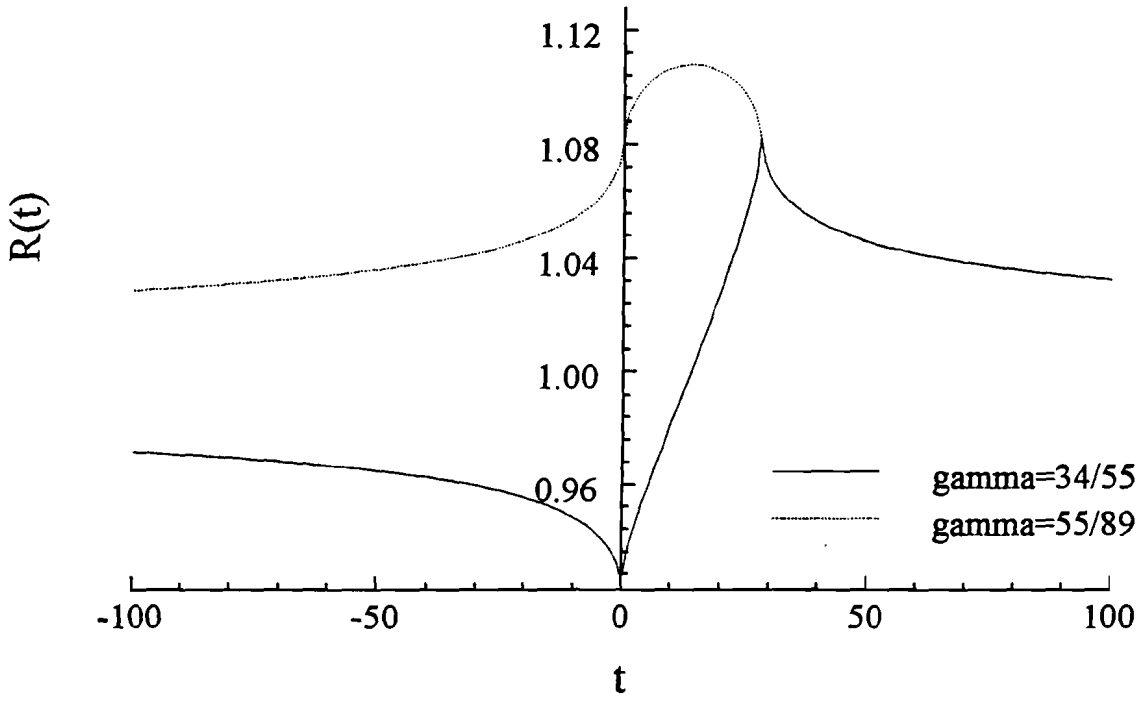


Figure 2b. R(t) function



Note: $g(t) = \exp[ht^{\gamma}]$, $R(t) = g(t)/g(t-A)$; $A = 28$, $h = .01$

Figure 4. Observed and Estimated NRRs,

United States 1960-1989

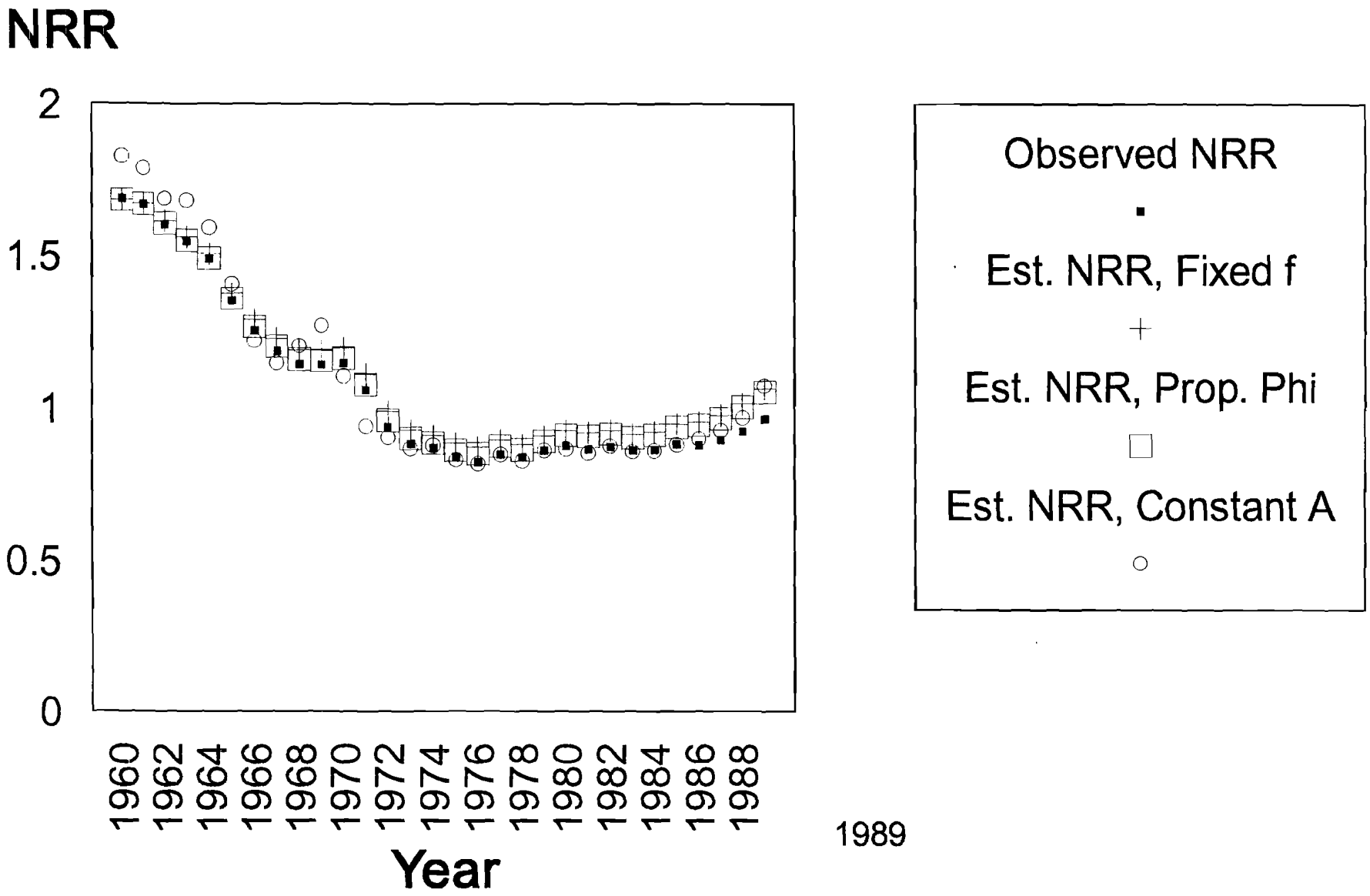
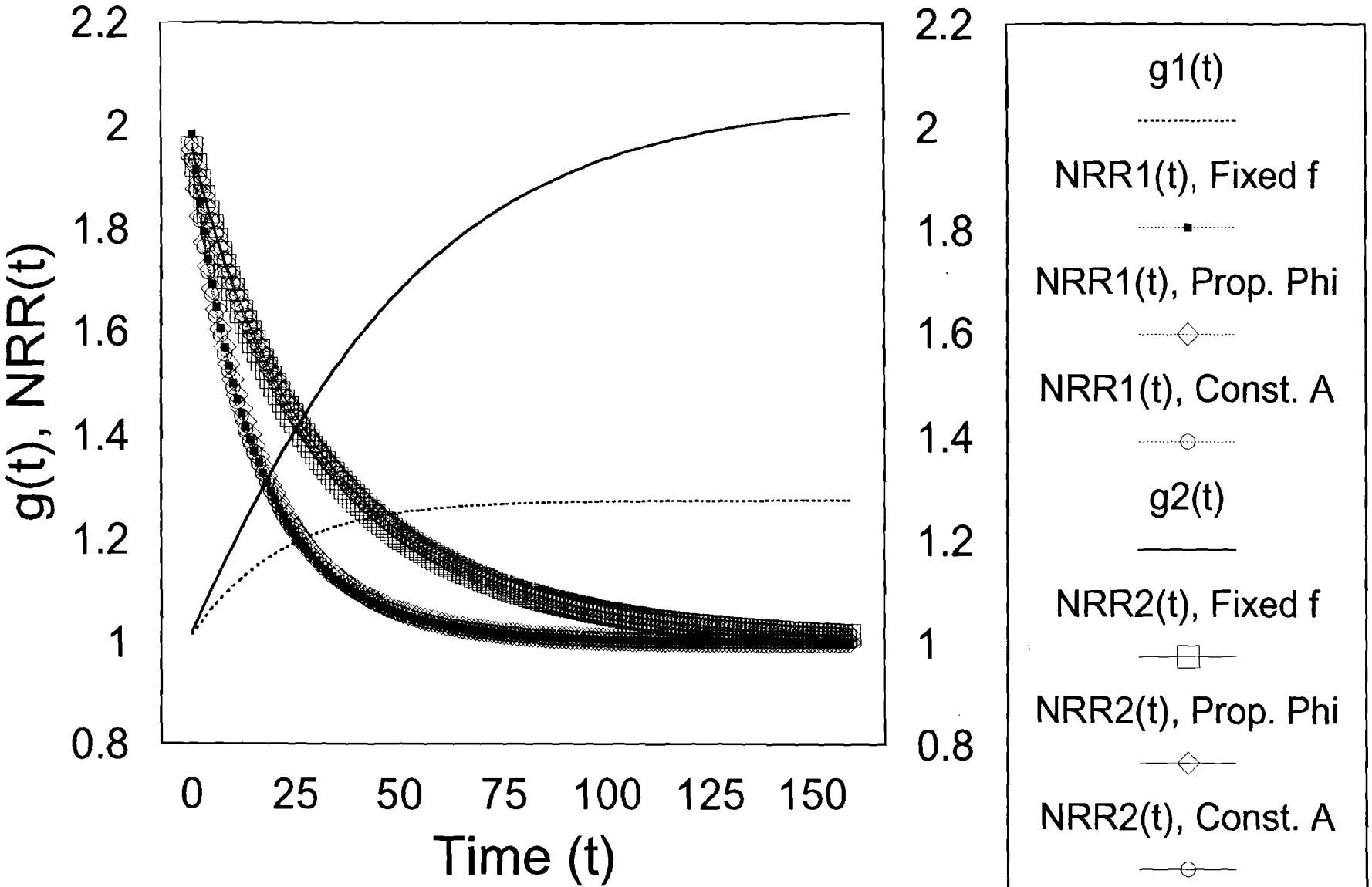


Figure 5. Double Exponential Birth Functions, g1(t) and g2(t) and Their Associated Net Reproduction Rates, NRR1(t) and NRR2(t).



NRR(0)=2 b1=.050 b2=.025